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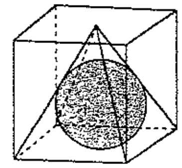
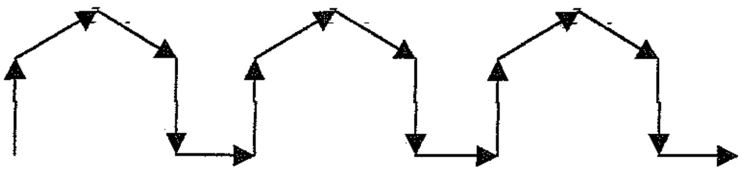


April 3, 2010
Round I: Arithmetic and Number Theory

1) Sketch the 2012^{th} arrow of the sequence:



2) Let the sum, difference, and the product of two numbers be S, D and P, respectively. The ratio S:D:P is 11:1:90. Find P.

3) What is the sum of the last two digits of the number named by $\left(4^3\right)^{2012}$?

1) ______________________________

2) ______________________________

3) ______________________________

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Round II
Algebra I

1) Solve: $|||x-1|-1|-1|=1$.

2) Solve for (x, y, z)

$$\begin{cases} \dfrac{3}{x}+\dfrac{1}{y}=61 \\ \dfrac{3}{y}-\dfrac{1}{z}=1 \\ \dfrac{1}{x}+\dfrac{3}{z}=26 \end{cases}$$

3) Solve: $\sqrt[3]{x^2+16x+64}-2\sqrt[3]{x+8}-24=0$.

1) ______________________________

2) ______________________________

3) ______________________________

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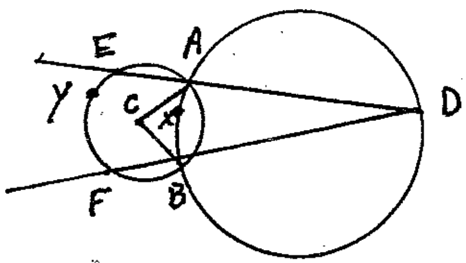
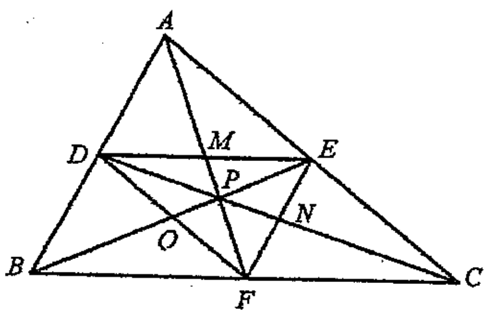
Round III
Geometry

1) In a right triangle, the hypotenuse has a length two more than the longer leg and a length one more than 5 times the shorter leg. What is the length of the hypotenuse?

2) Let C be the center of the circle. If $m\angle ACB = 40°$ and $m\widehat{EYF} = 60°$, find $m\widehat{AXB}$.



3) In ΔABC, the sum of the lengths of the medians $\overline{AF}, \overline{BE}$, and $\overline{CD}$ is 108.
Find the value of $MP + NP + OP$.



1) ______________________________

2) ______________________________

3) ______________________________

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Round IV
Algebra II

1) Find the domain of the function $f(x)=\sqrt{1-\ln x}$.

2) When $ax^{2012}-bx^{2011}+2010$ is divided by $x+1$, the remainder is 5. When the same expression is divided by $x-1$, the remainder is 15. Find the ordered pair (a, b).

3) For what values of <u>a</u> will the slope of the line containing the points (5, a) and (a, -2) be greater than or equal to 2?

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1) ______________________________

2) ______________________________

3) ______________________________

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Round V
Analytic Geometry

1) The line joining the vertices of a hyperbola is parallel to the x-axis and the center of the hyperbola is at (5, 3). The hyperbola passes through the origin and the point (7, 3). The equation of the hyperbola can be written in the form $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, where $a>0$ and $b>0$. Find the value of b.

2) Find coordinates of the endpoints of the major axis if the conic section with equation $3x^2+2y^2-18x+16y+58=0$.

3) An isosceles triangle with base endpoints (-4, -3) and (-3, 4) is inscribed into the circle $x^2+y^2=25$. What are the exact possible coordinates of the vertex angle of this triangle?

1) ____________________________

2) ____________________________

3) ____________________________

Round VI
Trigonometry, Complex Numbers

1) Solve for x, $0 \le x < 2\pi$

$$\frac{\tan x}{2\sin x} = -1.$$

2) Regular dodecagon ABCDEFGHIJKL has perimeter 108. Find the square of the distance between points F and H.

3) For what values of x, $0 \le x < 2\pi$ will $3 + 4\cos(3x)$ take its minimum value?

1) ________________________

2) ________________________

3) ________________________

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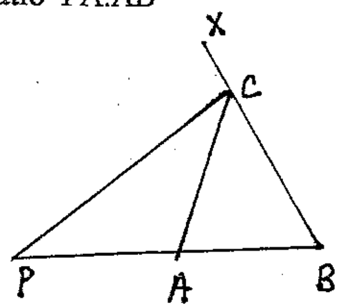
TEAM ROUND

1) The first 100 numbers of the sequence 9, 99, 999, 9999, ... are added together. If a b c d are the last four digits of the sum, find a b c d..

2)

Solve for x: $\frac{b+x}{b}+\frac{2x}{b+x}+\frac{x^2}{b^2-bx}=2$

3) In ΔABC (see the diagram) CA = 3 and CB = 4. PAB is a straight line and PC bisects angle ACX. Find the ratio PA:AB



4) Factor completely over the integers: $27x^3-y^3+3y^2-3y+1$.

5) Find the area of the polygon formed by the intersections of the graphs of

$$\begin{cases}(x-3)^2+(y+2)^2=289\\ y+19=\frac{1}{2}(x-3)^2\end{cases}$$

6) P(x) is a polynomial of least degree with integral coefficients (include the constant) with greatest common factor of 1. Find the sum of these coefficients (include the constant) if two of the roots of P(x) are $\frac{1}{2}+\frac{i\sqrt{7}}{3}$ and $3-2i$.

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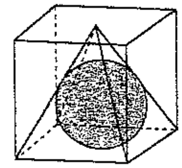

CSAML Answers
Host: E.O.Smith High School
April 3, 2012

Round I Arithmetic

1.

2. 270

3. ~~96~~ 15

Round II Algebra I

1. x = 4, x = -2, x = 2, x = 0

2. $\left(\frac{1}{20},1,\frac{1}{2}\right)$

3. x = 208, x = -72

Round III Geometry

1. 101

2. 20

3. 18

Round IV Algebra 2

1. $0 < x \le e$

2. (-2000, -5)

3. $a \in \left[\frac{8}{3},5\right)$

Round V Analytic Geometry

1. $\frac{2\sqrt{21}}{7}$

2. $\left(3,-4\pm\frac{\sqrt{2}}{2}\right)\left(3,-4\pm\frac{\sqrt{2}}{2}\right)$

3. $\left(\frac{7\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right),\left(-\frac{7\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

Round VI Trigonometry & Complex

1. $\frac{2\pi}{3},\frac{4\pi}{3}$

2. $81\left(2+\sqrt{3}\right)=162+81\sqrt{3}$

3. $\frac{\pi}{3},\pi,\frac{5\pi}{3}$

TEAM Round

1. 1010

2. $\frac{b}{3}$

3. 3:1

4. $\left(3x-y+1\right)\left(9x^2+3xy-3x+y^2-2y+1\right)$

5. 256

6. ~~-185~~ 296 ~~196~~

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April 3, 2012

Possible solutions:
Round I: Arithmetic and Number Theory

1. The pattern has 5 directional arrows, so $5n + R = 2012$.
   $5(402) + 2 = 2012$, so the 2^{nd} leg is the 2012^{th}.

2. S:D:P=11:1:90 or $(a + b):(a - b): ab = 11:1:90$

$$\begin{cases} a+b=11x \\ a-b=x \\ ab=90x \end{cases} \rightarrow a+b=11a-11b \rightarrow a=\frac{6b}{5}, x=\frac{b}{5}$$

$$\frac{6b^2}{5}=(90)\left(\frac{b}{5}\right) \rightarrow b=15, a=18 \rightarrow ab=270$$

3. The last 2 digits of 4^n has a 10 cycle repeat:

<u>4</u>, 16, 64, 256, 1024, 4096, 16384, 65536, 262144, 104856, 41943<u>04</u>, ..So, $\frac{6036}{10}=603+\frac{6}{10}$.
Now the last 2 digits of our problem are 96, and the sum of these two digits is <u>15</u>.

Round II. Algebra I

1.

$$||x-1|-1|-1|=1 \rightarrow ||x-1|-1|-1=1 \rightarrow ||x-1|-1|=2 \text{ or } ||x-1|-1|=0$$

$$||x-1|-1|=2 \rightarrow |x-1|-1=\pm 2 \rightarrow |x-1|=3 \text{ or } |x-1|=-1$$

$$|x-1|=3 \rightarrow x-1=\pm 3 \rightarrow x=4 \text{ or } x=-2$$

$$||x-1|-1|=0 \rightarrow |x-1|-1=0 \rightarrow |x-1|=1 \rightarrow x-1=\pm 1 \rightarrow x=2 \text{ or } x=0$$

2.

$$\begin{cases} 3A+B+0=61 \\ 0+3B-C=1 \\ A+0+3C=26 \end{cases} \rightarrow \begin{cases} 0+9B-3C=3 \\ A+0+3C=26 \end{cases} \rightarrow A+9B=29$$

$$\begin{cases} 3A+B=61 \\ A+9B=29 \end{cases} \rightarrow \begin{cases} 3A+B=61 \\ 3A+27B=87 \end{cases} \rightarrow 26B=26 \rightarrow =1$$

$$3A+B=61 \rightarrow 3A=60 \rightarrow A=20$$

$$A+0+3C=26 \rightarrow 3C=6 \rightarrow C=2$$

$$(A,B,C)=(20,1,2) \rightarrow (x,y,z)=\left(\frac{1}{20},1,\frac{1}{2}\right)$$

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3. Rewrite the problem using fractional exponents.

$$(x+8)^{2/3}-2(x+8)^{1/3}-24=0$$

Let $A=(x+8)^{1/3} \to A^2-2A-24=0$

$$(A+4)(A-6)=0$$

$$\sqrt[3]{x+8}=-4 \to x+8=-64 \to x=-72$$

$$\sqrt[3]{x+8}=6 \to x+8=216 \to x=208$$

Round III: Geometry

1. I chose x = the hypotenuse. Now, one leg is (x – 2) and the other leg is (x-1)/5. By the Pythagorean Theorem:

$$\frac{(x-1)^2}{25}+(x-2)^2=x^2 \to \frac{x^2-2x+1}{25}+x^2-4x+4=x^2$$

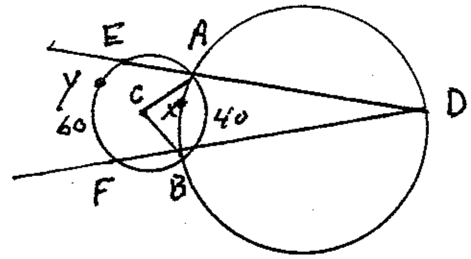
$$x^2-2x+1=100x-100 \to x^2-102x+101=0$$

$$(x-1)(x-101)=0 \to x=101$$

2.

$$m\angle D=\frac{60-40}{2}=10$$

$$m\angle D=\frac{1}{2}m\widehat{AXB} \to m\widehat{AXB}=20$$



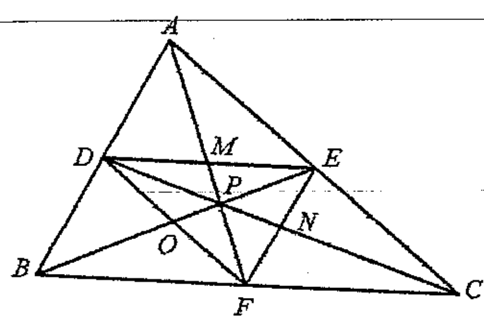
3.

Since M is the midpoint of $\overline{DF}$, then $MF=\frac{1}{2}AF$ and since $\overline{FM}$ is a median of ΔDEF,

then $MP=\frac{1}{3}MF \to MP=\frac{1}{6}AF$.

Likewise, $NP=\frac{1}{6}CD$ and $OP=\frac{1}{6}BE$. So $MP+NP+OP=\frac{1}{6}(108)=18$

Round IV: Algebra II



1. $0<\ln x\le 1$, $\ln x=\log_e x=1 \to x=e^1$

$0<x\le e$

2. $\begin{cases} P(-1) = a + b + 2010 = 5 \\ P(1) = a - b + 2010 = 15 \end{cases} \to 2a + 4020 = 20 \to a = -2000, b = -5 \to (-2000, -5)$

3.

$$m = \frac{a+2}{5-a} \geq 2 \to \frac{a+2}{5-a} - 2 \geq 0$$

$$\frac{a+2}{5-a} - \frac{2(5-a)}{(5-a)} \geq 0 \to \frac{3a-8}{5-a} \geq 0$$

$$\frac{8}{3} \leq a < 5 \text{ or } \left[\frac{8}{3}, 5\right)$$

Round V: Analytic Geometry

1. Substitute each given point into the equation and find <u>b</u>.
Given:

$$\frac{(x-5)^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$$

$$\text{with } (0,0)\ \frac{25}{a^2} + \frac{9}{b^2} = 1$$

$$\text{with } (7,3)\ \frac{4}{a^2} + 0 = 1 \to a^2 = 4$$

$$\text{Now, } \frac{25}{4} + \frac{9}{b^2} = 1 \to b^2 = \frac{36}{21} \to b = \frac{2\sqrt{21}}{7}$$

2. First find the center of the ellipse and the value of <u>a</u>.

$$3x^2 - 18x + \_\_\_\_ + 2y^2 + 16y + \_\_\_\_ = -58$$

$$3(x^2 - 6x + 9) + 2(y^2 + 8y + 16) = -58 + 3(9) + 2(16)$$

$$3(x-3)^2 + 2(y+4)^2 = 1 \to \frac{(x-3)^2}{1/3} + \frac{(y+4)^2}{1/2} = 1$$

Notice that $\frac{1}{2}$ is greater than $\frac{1}{3}$, so the major axis is parallel to the y-axis and the ends of the major axis are at $\left(3, -4 \pm \frac{\sqrt{2}}{2}\right)$.

3. The distance from the point (x, y) to (-4, -3) is equal to the distance from the point (x, y) to (-3, 4). So,

$$\sqrt{(x+3)^2+(y-4)^2}=\sqrt{(x+4)^2+(y+3)^2}$$
$$x^2+6x+9+y^2-8y+16=x^2+8x+16+y^2+6y+9$$
$$-2x=14y\to x=-7y$$
$$x^2+y^2=25\to 49y^2+y^2=25\to y=\pm\frac{\sqrt{2}}{2}\to x=\mp\frac{7\sqrt{2}}{2}$$
$$\left(\frac{7\sqrt{2}}{2},\frac{-\sqrt{2}}{2}\right),\left(\frac{-7\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$$

Round VI – Trigonometry, Complex numbers.

1. $\frac{\tan x}{2\sin x}=-1\to\frac{1}{\cos x}=-2\to\cos x=-\frac{1}{2}\to\to x=\frac{2\pi}{3},\frac{4\pi}{3}$

2. When you joint the vertices F and H, you form an isosceles triangle. Notice that a dodecagon has an interior angle of $150°$. Now you can find the square of the distance FH since it is the 3rd side of the triangle. Using the law of cosines:

$$x^2=9^2+9^2-2(9)(9)\cos150°$$
$$x^2=162-2(81)\left(-\frac{\sqrt{3}}{2}\right)=162+82\sqrt{3}$$

3. The cos 3x is at a minimum value when $3x=\pi+2\pi n$ for n = 0, 1, 2, ...

$$x=\frac{\pi}{3}+\frac{2\pi n}{3}$$
$$0\le\frac{\pi}{3}+\frac{2\pi n}{3}<2\pi$$
$$\frac{\pi}{3},\pi,\frac{5\pi}{3}$$

Team Round

1. There are 100 9's in the units position, 99 9's in the tens position, 98 9's in the hundreds place and 97 9's for the thousands place. The sums is 900 + 9(990) + 9(9800) + 9(97000). Add these values up to the 4 necessary digits. 1010

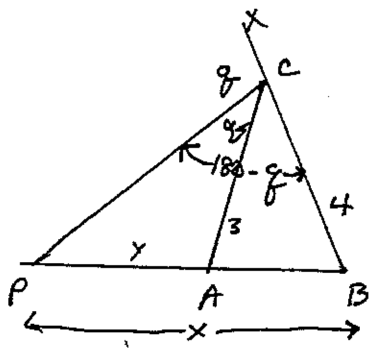
2. Multiply by the LCM = b(b + x)(b – x) and get:

$(b+x)(b+x)(b-x)+2x(b)(b-x)+x^2(b+x)=2b(b+x)(b-x);\quad b\neq \pm b, b\neq 0$

$b^3-bx^2+b^2x-x^3+2b^2x-2bx^2+bx^2+x^3=2b^3-2bx^2$

$3b^2x=b^3\rightarrow x=\dfrac{b}{3}$

3. Using trigonometry:



$\dfrac{4}{\sin\theta}=\dfrac{x}{\sin(180-q)}=\dfrac{x}{\sin q}$ and $\dfrac{3}{\sin\theta}=\dfrac{y}{\sin q}$

$\dfrac{3}{y\sin\theta}=\dfrac{4}{x\sin\theta}\rightarrow\dfrac{x}{y}=\dfrac{4}{3}\rightarrow\dfrac{y}{x-y}=\dfrac{3}{1}$

Using Geometry:

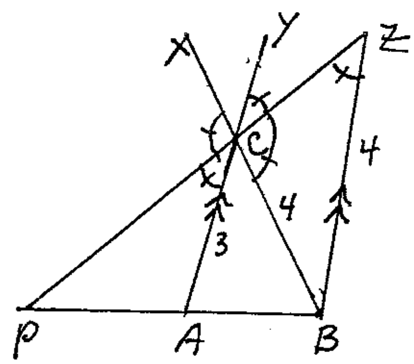
Draw

$BZ\parallel AC, \angle XCP\cong\angle PCA\cong\angle Z$

$m\angle YCZ=m\angle ZCB=q$

$\Delta PAC\approx\Delta PBZ\rightarrow\dfrac{3}{AP}=\dfrac{4}{PB}\rightarrow\dfrac{PA}{PB}=\dfrac{3}{4}$

$\dfrac{PA}{AB}=\dfrac{3}{1}$



4.

$27x^3-y^3+3y^2-3y+1$

$(3x)^3-(y-1)^3$

$(3y-y+1)(9x^2+3xy-3x+y^2-2y+1)$

5.

$\begin{cases}(x-3)^2+(y+2)^2=17^2\\ y+19=\dfrac{1}{2}(x-3)^2\end{cases}\rightarrow(x-3)^2=17^2-(y+2)^2$

$2(y+19)=17^2-(y+2)^2$

$2(y+19)+(y+2)^2=17^2$

(CONT.)

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$2y+38+y^2+4y+4=289$

$y^2+6y+\_\_\_\_=289-42+\_\_\_\_$

$y^2+6y+9=247+9=256=16^2$

$(y+3)^2=16^2 \to (y+3)=\pm 16 \to y=\to 13,-19$

$(x-3)^2=2(13)+38 \to (x-3)=\pm 8 \to x=11,-5$

$(11,13),(-5,13)$

$(x-3)^2=2(-19)+38 \to x=3 \to (3,-19)$

A triangle is formed by these three points. The base of the triangle (11, 13) and (-5, 13) has a length of 16 and the height is [13 – (-19)] or 32, so the area is 16(32)/2 = 256.

6. The polynomial has integral coefficients so the least degree must be 4 with roots $\left(\frac{1}{2}+\frac{i\sqrt{7}}{3}\right),\left(\frac{1}{2}-\frac{i\sqrt{7}}{3}\right),(3-2i),(3+2i)$. Notice that we have pairs of roots of the form $a+bi$ and $a-bi$. If we multiply $(x-a-bi)(x-a+bi) \to x^2-2ax+\left(a^2+b^2\right)$. We can use this form with our given roots to get

$a=\frac{1}{2}, b=\frac{\sqrt{7}}{3} \to x^2-2\left(\frac{1}{2}\right)x+\left(\frac{1}{4}+\frac{7}{9}\right) \to 36x^2-36x+37=0$ and when

$a=3, b=-2 \to x^2-2(3)x+(9+4) \to x^2-6x+13$. Now multiply these to find the coefficients.

$\left(x^2-6x+13\right)\left(36x^2-36x+37\right) \to$

$(36x^4-36x^3+37x^2)+(-216x^3+216x^2-222x)+(468x^2-468x+481)$

$36-36+37-216+216-222+468-468+481=296$